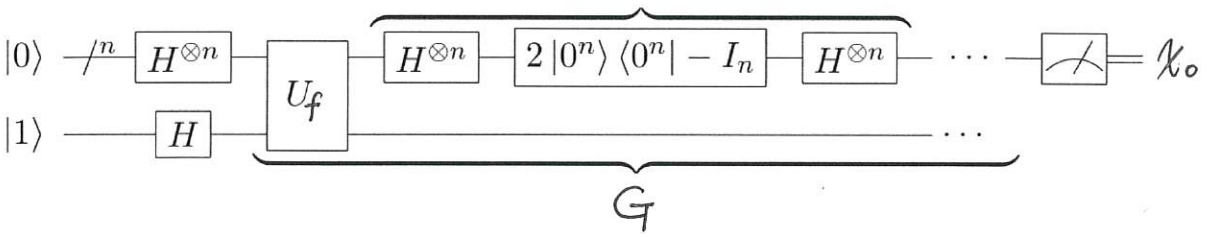
Grover algorithm

$$2|\psi\rangle\langle\psi| - I$$
$$\|$$
Grover diffusion operator



Repeat $k = O(\sqrt{N})$ times

Algorithm: Quantum search

Inputs: (1) a black box oracle $U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$

(2) $f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases} \qquad 0 \leq x < 2^n$

Outputs: x_0.

Runtime: $O(\sqrt{2^n})$ operations. Succeeds with probability $O(1)$.

Procedure:

1. $|0\rangle^{\otimes n}|1\rangle$
2. $\xrightarrow{H^{\otimes n}+H} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$
3. $\xrightarrow{G^k} \left[(2|\psi\rangle\langle\psi| - I)\, U_f\right]^k \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$

   $\approx |x_0\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$

   $k = \left[\frac{\pi}{4}\sqrt{N} - \frac{1}{2}\right]$
4. $\rightarrow x_0$ measure the first n qubits

No.

說明

(2) $H^{\otimes n}|0^n\rangle \longrightarrow |\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$

$\begin{cases} N = 2^n & = 2^3 \\ A = \{x \mid f(x)=1,\ 0 \le x < N\} & = \{011\} \\ |A| = a & = 1 \end{cases}$

$$= \sqrt{\frac{N-a}{N}} \underbrace{\frac{1}{\sqrt{N-a}} \sum_{x \notin A} |x\rangle} + \sqrt{\frac{a}{N}} \underbrace{\frac{1}{\sqrt{a}} \sum_{x \in A} |x\rangle}$$

$$= \cos\theta \underbrace{|\alpha\rangle} + \sin\theta \underbrace{|\beta\rangle}$$

$\{|\alpha\rangle, |\beta\rangle\}$ orthonormal basis

(3)

① $U_f : |xy\rangle \longrightarrow |x,\ y \oplus f(x)\rangle$

$$|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \longrightarrow (-1)^{f(x)} |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \longrightarrow \left(\cos\theta |\alpha\rangle - \sin\theta |\beta\rangle\right) \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

② $\left(2|0^n\rangle\langle 0^n| - I_n\right)|x\rangle \longrightarrow \begin{cases} |x\rangle & x = 0^n \\ -|x\rangle & x \neq 0^n \end{cases}$

③ $H^{\otimes n}\left(2|0^n\rangle\langle 0^n| - I_n\right) H^{\otimes n} = 2|\psi\rangle\langle\psi| - I_n$

④ $\left(2\underbrace{|\psi\rangle\langle\psi|} - I\right) \underbrace{\sum_{k=0}^{N-1} a_k |k\rangle}$ $\qquad B = \{|0\rangle, |1\rangle, \cdots, |N-1\rangle\}$

$$= \left(\frac{2}{N} \underbrace{\sum_{k=0}^{N-1} a_k}\right) \sum_{k=0}^{N-1} |k\rangle - \sum_{k=0}^{N-1} a_k |k\rangle$$

$$= \sum_{k=0}^{N-1} (2\bar{a} - a_k) |k\rangle \qquad \bar{a} = \frac{1}{N} \sum_{k=0}^{N-1} a_k \ \text{(mean)}$$

Note: (甲) $\frac{1}{N} \sum_{k=0}^{N-1} (2\bar{a} - a_k) = 2\bar{a} - \bar{a} = \bar{a}$

(乙) $\sum_{k=0}^{N-1} (2\bar{a} - a_k)^2 = \sum_{k=0}^{N-1} (4\bar{a}^2 - 4\bar{a} a_k + a_k^2)$

$$= 4N\bar{a}^2 - 4\bar{a} N \bar{a} + \sum_{K=0}^{N-1} a_k^2$$

$$= 1$$

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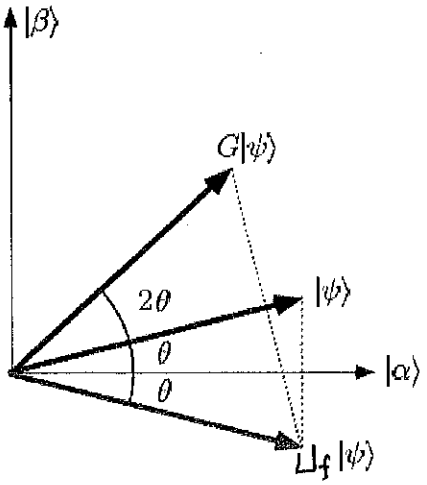
⑤ $G_T = (2|\psi\rangle\langle\psi| - I)\sqcup_f$ $\quad B' = \{|\alpha\rangle, |\beta\rangle\}$

$$= \left(2\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}[\cos\theta\ \sin\theta] - \begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$= \begin{bmatrix}2\cos^2\theta-1 & 2\cos\theta\sin\theta\\ 2\cos\theta\sin\theta & 2\sin^2\theta-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$= \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$= \begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}$$



⑥ $G|\psi\rangle = \cos 3\theta\,|\alpha\rangle + \sin 3\theta\,|\beta\rangle$

$G^k|\psi\rangle = \cos(2k+1)\theta\,|\alpha\rangle + \sin(2k+1)\theta\,|\beta\rangle$

$k = \left[\frac{\pi}{4}\sqrt{N} - \frac{1}{2}\right] \Rightarrow \sin(2k+1)\theta = \sin\left(2\left(\frac{\pi}{4}\sqrt{N} - \frac{1}{2} + \varepsilon\right)+1\right)\theta$ $\quad |\varepsilon| < \frac{1}{2}$

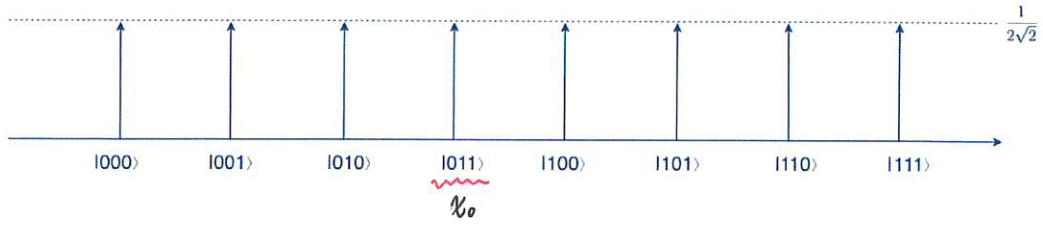
$= \sin\left(\frac{\pi}{2}\sqrt{N} + 2\varepsilon\right)\theta$

$\approx \sin\frac{\pi}{2} = 1$

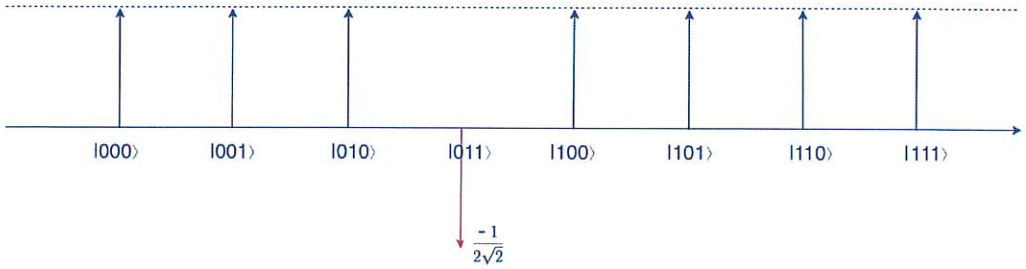
$\sin\theta = \sqrt{\frac{1}{N}} \approx \theta$ (N 很大)

(1)

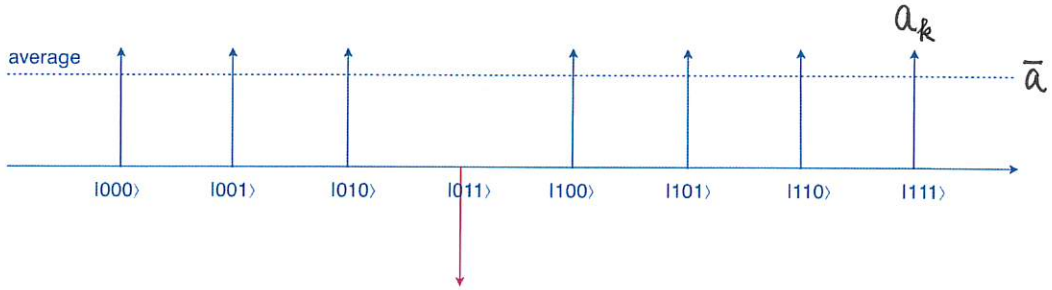
$$H^3|000\rangle = \frac{1}{2\sqrt{2}}|000\rangle + \frac{1}{2\sqrt{2}}|001\rangle + \ldots + \frac{1}{2\sqrt{2}}|111\rangle = \frac{1}{2\sqrt{2}}\sum_{x=0}^{7}|x\rangle = |\psi\rangle$$



(2) U_f

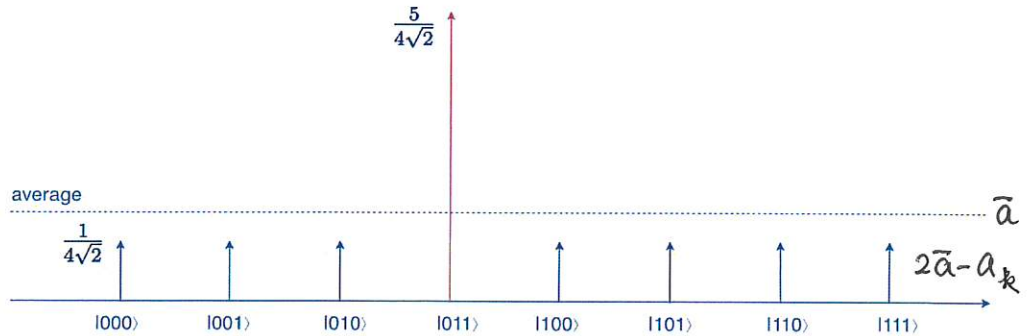


(3)



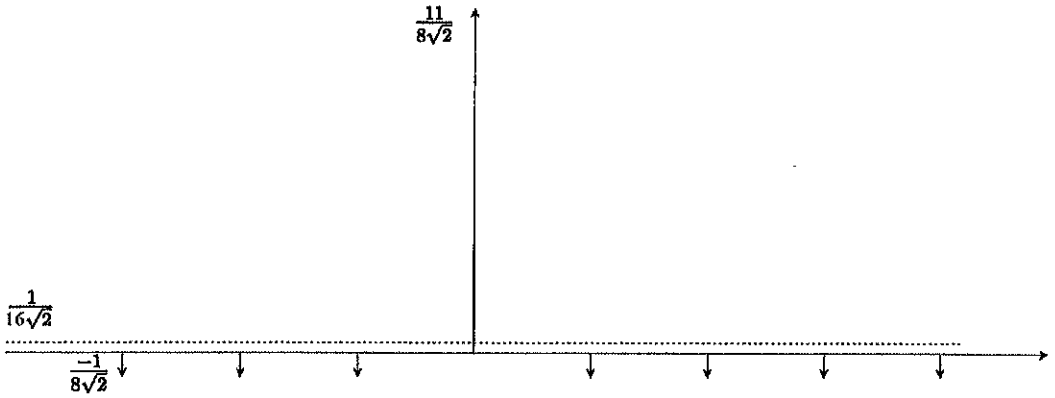
(4) $2|\Psi\rangle\langle\Psi| - I$

$$|x\rangle = \frac{1}{4\sqrt{2}}|000\rangle + \frac{1}{4\sqrt{2}}|001\rangle + \frac{1}{4\sqrt{2}}|010\rangle + \frac{5}{4\sqrt{2}}|011\rangle + \ldots + \frac{1}{4\sqrt{2}}|111\rangle$$



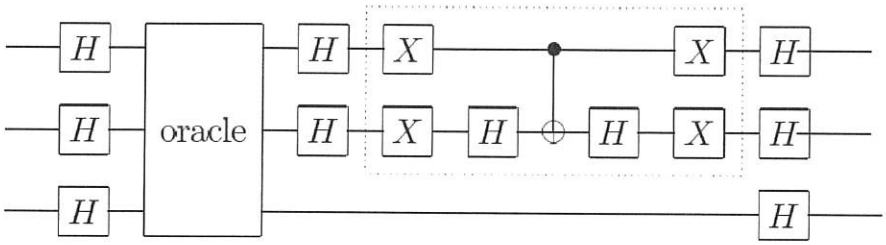
(5) G_1

$$|x\rangle = -\frac{1}{8\sqrt{2}}|000\rangle - \frac{1}{8\sqrt{2}}|001\rangle - \frac{1}{8\sqrt{2}}|010\rangle + \frac{11}{8\sqrt{2}}|011\rangle - \ldots - \frac{1}{8\sqrt{2}}|111\rangle$$



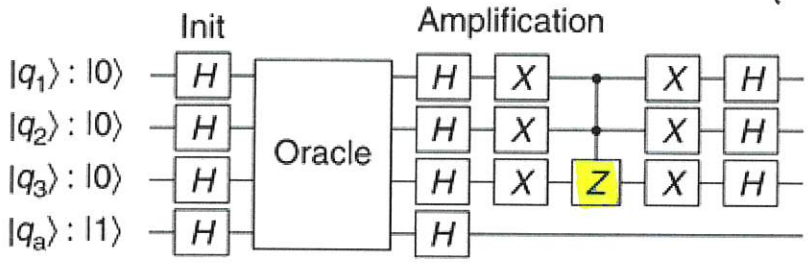
Quantumn Circuit

(A)

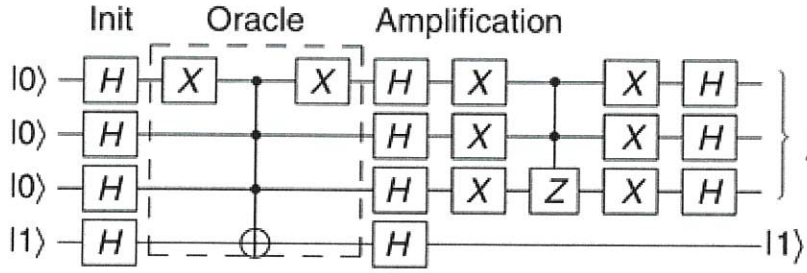


$\Downarrow$ $HXH = Z$: $\begin{cases} |0\rangle \to |0\rangle \\ |1\rangle \to -|1\rangle \end{cases}$

(B)



(C)



$\frac{5}{4\sqrt{2}}|011\rangle + \frac{1}{4\sqrt{2}}\Sigma_{x\neq 011}|x\rangle$

(1) $f(x) = \bar{x}_1 \wedge x_2 \wedge x_3 = \begin{cases} 1 & x = 011 \\ 0 & \neq \end{cases}$

$U_f|x,y\rangle = |x, f(x)\oplus y\rangle = |x_1x_2x_3, \bar{x}_1 \wedge x_2 \wedge x_3 \oplus y\rangle$

(2) $U_0 = (2|0\rangle\langle 0| - I)$ $\quad |x\rangle = \begin{cases} |x\rangle & x = 0 \\ -|x\rangle & x \neq 0 \end{cases}$

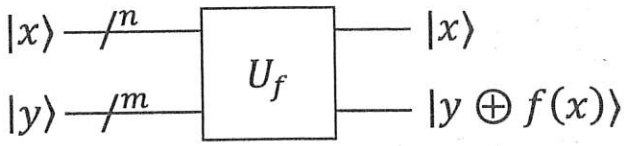
(甲) $-U_0|x\rangle = (-1)^{\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3}|x\rangle = \begin{cases} -|x\rangle & x = 0 \\ |x\rangle & x \neq 0 \end{cases}$

(乙) $|g_3\rangle$: $|x_3\rangle \to |\bar{x}_3\rangle \to \begin{cases} |\bar{x}_3\rangle & x_1x_2 \neq 00 \\ |\bar{x}_3\rangle & x_1x_2x_3 = 001 \\ -|\bar{x}_3\rangle & \text{〃} = 000 \end{cases}$

$= \begin{cases} |\bar{x}_3\rangle & x \neq 000 \\ -|\bar{x}_3\rangle & x = 000 \end{cases} \to \begin{cases} |x_3\rangle & x \neq 0 \\ -|x_3\rangle & x = 0 \end{cases}$

$|g_1g_2g_3\rangle$: $|x_1x_2x_3\rangle \to \begin{cases} -|x_1x_2x_3\rangle & x = 0 \\ |x_1x_2x_3\rangle & x \neq 0 \end{cases}$

實作 Black box U_f



- Given $f: \mathbb{B}^n \to \mathbb{B}^m$

  實作 $U_f: \mathcal{H}_{2^{n+m}} \to \mathcal{H}_{2^{n+m}}$ quantumn circuit

  $: |x,y\rangle \mapsto |x, y \oplus f(x)\rangle$

說明:

(1) $U_f: \mathbb{E} = \{ |x,y\rangle \mid \begin{matrix} x \in \mathbb{B}^n \\ y \in \mathbb{B}^m \end{matrix} \} \to \mathbb{E}$ (basis → basis)

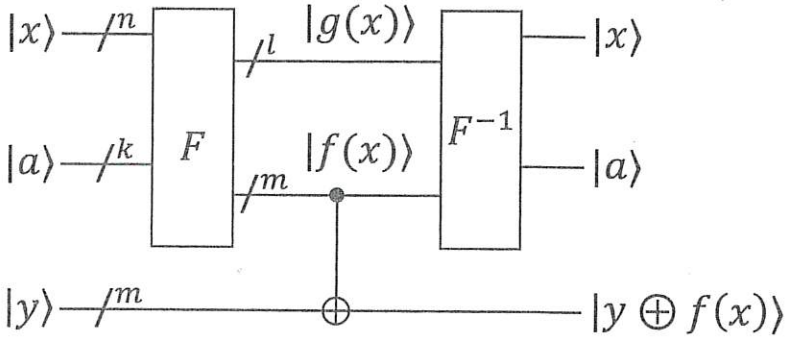
(2) $U_f^2 |x,y\rangle \to U_f |x, y \oplus f(x)\rangle \to |x, (y \oplus f(x)) \oplus f(x)\rangle = |x,y\rangle$

(3) U_f : involution $(U_f^2 = I)$

: unitary

: Hermitian

(4) can implement F (即 $f(x)$) (CCNOT) ($|a\rangle$ 補助位)



- Goal of Quantumn algorithms
  - Given U_f: $|x,y\rangle \to |x, y \oplus f(x)\rangle$ where f promises property P.
  - design a quantumn algorithm to find P.